

Splitting Madsen-Tillmann spectra II. The Steinberg idempotents and Whitehead conjecture

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Abstract

We show that, at the prime $p = 2$, the spectrum $\Sigma^{-n}D(n)$ splits off the Madsen-Tillmann spectrum $MTO(n) = BO(n)^{-\gamma_n}$ which is compatible with the classic splitting of $M(n)$ off $BO(n)_+$. For $n = 2$, together with our previous splitting result on Madsen-Tillmann spectra, this shows that $MTO(2)$ is homotopy equivalent to $BSO(3)_+ \vee \Sigma^{-2}D(2)$.

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1 Introduction

The Madsen-Tillmann spectrum $MTO(n)$ is defined to be the Thom spectrum of the virtual bundle $-\gamma_n \rightarrow BO(n)$ where γ_n is the universal n -plane bundle. It is known that these spectra filter the spectrum MO , i. e. there is a sequence

$$S^0 = MTO(0) \rightarrow \Sigma MTO(1) \rightarrow \cdots \rightarrow \Sigma^{n-1} MTO(n-1) \xrightarrow{\iota_n} \Sigma^n MTO(n) \rightarrow \cdots \quad (1)$$

where ι_n is induced by the inclusion $O(n-1) \subset O(n)$, with the property $\text{colim } \Sigma^n MTO(n) \cong MO$. Furthermore the cofiber of the successive stages is homotopy equivalent to $BO(n)_+$, that is, we have a cofiber sequence [2]

$$\cdots \rightarrow \Sigma^{-1} MTO(n-1) \rightarrow MTO(n) \xrightarrow{p_n} BO(n)_+ \rightarrow MTO(n-1) \rightarrow \cdots \quad (2)$$

where p_n is the map induced by the “embedding” of (-1) times the canonical bundle into the 0-dimensional trivial bundle. In other words, the spectrum MO can be built up from pieces $BO(n)_+$.

We have shown that localised away from 2, $MTO(2n) \simeq BO(2n)_+$ and $MTO(2n+1) \simeq *$ for all $n \geq 0$ [4, Theorem 1.1.B]. Thus the study of $MTO(n)$ ’s become essentially 2-local problems. Therefore we shall work at the prime $p = 2$. For technical reasons, we will rather work with 2-completed spectra instead of 2-local spectra, and in the rest of the paper we identify a spectrum with its 2-completion. Since our main application concerns the mod 2 cohomology of associated infinite loop spaces, by passing to 2-completion, no information will be lost. Throughout the paper homology and cohomology are taken with $\mathbb{Z}/2$ coefficients. As we work essentially with spectra, we identify spaces with its suspension spectra. In the literature, sometimes a space X is identified with the suspension spectra of the space with added basepoint X_+ , which explains a notational discrepancy the reader may find between the current paper and results we quote.

At the prime 2, Randal-Williams computed $H_*(\Omega^\infty MTO(i))$ for $i = 1$ and 2, [12, Theorems A and B]. Combining the two theorems, we get an exact sequence of Hopf algebras

$$H_*(Q_0 BO(2)_+) \rightarrow H_*(Q_0 BO(1)_+) \rightarrow H_*(Q_0 BO(0)_+) \rightarrow \mathbb{Z}/2 \quad (3)$$

where the (Hopf) kernel of the first two maps are isomorphic to $H_*(\Omega^\infty MTO(i))$ for $i = 2$ and 1 respectively. Thus a natural question to ask was whether if this exact sequence could be extended further to the left with $H_*(\Omega^\infty MTO(i))$ isomorphic to the kernel of each stage. We showed that this was impossible in [4]. So a new question to ask, then, is to what extent we can generalize [12, Theorems A and B].

This question leads to a search for another sequence of spectra with the beginning as in (1). It turns out that there indeed is such a sequence, well-known to stable homotopy theorists, that is:

$$S^0 = D(0) \rightarrow D(1) \rightarrow \cdots \rightarrow D(n-1) \xrightarrow{\iota_n} D(n) \rightarrow \cdots . \quad (4)$$

These spectra realizes the length filtration of the Steenrod algebra, that is, we have

$$H^*(D(n)) \cong \mathcal{A}/G_n \text{ where } G_n \text{ is the span of } Sq^I, I \text{ admissible, } l(I) > n .$$

We note that G_n happens to be a left \mathcal{A} -ideal, so that this isomorphism is as \mathcal{A} modules. They originally were defined using the symmetric powers ([9, Proposition 4.3]), and we have colim $D(n) = H\mathbb{Z}/2$, and cofibrations

$$\rightarrow \Sigma^{-n-1}M(n) \rightarrow D(n-1) \rightarrow D(n) \rightarrow \Sigma^{-n}M(n) \rightarrow \quad (5)$$

The spectrum $M(n)$ is defined to be the cofibre of the map $\Sigma^{-n}D(n-1) \rightarrow \Sigma^{-n}D(n)$. Thus one can say that $H\mathbb{Z}/2$ can be built up of $M(n)$'s. These building blocks can be described alternatively as follows.

The spectra $BO(1)_+^{\times n}$ admit a natural (left) $Gl_n(\mathbb{Z}/2)$ action. Thus the Steinberg idempotent $e_n \in \mathbb{Z}/2[Gl_n(\mathbb{Z}/2)]$ gives rise to a splitting of $BO(1)_+^{\times n}$ and we have $M(n) \simeq e_n BO(1)_+^{\times n}$ [9, Theorem 5.1]. Moreover, through the Becker-Goettlieb transfer map, this splitting gives rise to a splitting of $M(n)$ off $BO(n)_+$ (see Mitchell and Priddy's paper [9] for more details and for the odd primary splitting results as well).

Therefore we have a construction of MO with $BO(n)_+$'s as building blocks, a construction of $H\mathbb{Z}/2$ with $M(n)$'s as building blocks. Furthermore $H\mathbb{Z}/2$ and $M(n)$'s split off respectively MO and $BO(n)_+$'s. It is then natural to ask whether one can split intermediate stages. The purpose of this paper is to answer affirmatively to this question, and discuss some consequences, including an answer to the question on generalization of the exact sequence (3). More detailed statements are given in the next section.

The paper is organized as follows. In section 2, we summarize our results. In section 3, we recall relevant results from [5] and construct a map from F_*Y to F_*X . In section 4 we use Takayasu's results [13] to construct a map going the other way round, and show that we indeed have a splitting. In section 5 we discuss the consequences in homology of infinite loop spaces.

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2 Statement of results

To give precise statements, we start with some defintions.

Definition 2.1. *i). A filtered spectrum is a sequence of spectra F_*X*

$$F_0X \rightarrow F_1X \rightarrow \cdots \rightarrow F_nX \rightarrow F_{n+1}X \rightarrow \cdots \quad (6)$$

with a homotopy equivalence $\text{hocolim} F_nX \simeq X$.

ii). A map of filtered spectra (f_) from F_*X to F_*Y is a collection of maps $f_n: F_nX \rightarrow F_nY$ that makes the squares*

$$\begin{array}{ccc} F_nX & \longrightarrow & F_{n+1}X \\ \downarrow & & \downarrow \\ F_nY & \longrightarrow & F_{n+1}Y \end{array}$$

commutative.

- iii). Two filtered spectrum F_*X and F_*Y are said to be equivalent if there is a map f_* from F_*X to F_*Y such that f_n 's are homotopy equivalence for all n .
- iv). We say that F_*X splits off F_*Y if there are maps f_* from F_*X to F_*Y and g_* from F_*Y to F_*X such that $H_*(f_n \circ g_n) = id$.

Our main result then reads as follows.

Theorem 2.2. Define filtered spectra F_*X and F_*Y by $F_nX = D(n)$, $F_nY = \Sigma^n MTO(n)$. Then F_*X splits off F_*Y .

As our proof doesn't depend on the decomposition of MO ([15, Theorem 2.10], [14, Theorem 2]), we obtain a "new" proof of the splitting of $H\mathbb{Z}/2$ off MTO . However, our method doesn't allow us to obtain information on other summands.

An immediate consequence is the following.

Corollary 2.3. $H_*(\Omega^\infty \Sigma^{-n} D(n))$ splits off $H_*(\Omega^\infty MTO(n))$ as a Hopf algebra.

Thus the "correct way to extend" the exact sequence (3) is just the following standard fact

Proposition 2.4 ([5]). The following sequence of Hopf algebras is exact.

$$\cdots \rightarrow H_*(\Omega^\infty M(n)) \rightarrow H_*(\Omega^\infty M(n-1)) \rightarrow \cdots \rightarrow H_*(\Omega^\infty M(2)) \rightarrow H_*(Q_0 B\mathbb{Z}/2_+) \rightarrow H_*(Q_0 S^0) \rightarrow \mathbb{Z}/2$$

Furthermore the image of $H_*(\Omega^\infty M(n)) \rightarrow H_*(\Omega^\infty M(n-1))$ is isomorphic to $H_*(\Omega_0^\infty D(n-1))$.

As $D(0) \cong S^0$, $\Sigma^{-1} D(1) \cong MTO(1)$ [9, Proposition 4.4], and $M(1) \cong BO(1)_+$, combined with the $n = 2$ case of Theorem 2.2, we recover Theorems A and B of [12].

Of course, the cohomology being dual of homology, the exact sequences above give some information on certain characteristic classes. More precisely, recall from [12, 4]

Definition 2.5. A universally defined characteristic class in $H^*(\Omega_0^\infty MTO(n))$ is a element in the subalgebra generated by the image of $H^*(BO(n)) \rightarrow H^*(Q_0 BO(n)) \rightarrow H^*(\Omega_0^\infty MTO(n))$. We denote $\mu_{i_1, \dots, i_n} = (\Omega^\infty p_n)^*(\sigma^{\infty*}(\sigma_1^{i_1} \dots \sigma_n^{i_n}))$ where $H^*(BO(n)) \cong \mathbb{Z}/2[\sigma_1, \dots, \sigma_n]$.

In [4], we used the summand $BSO(2n+1)_+$ that split off $MTO(2n)$ to show that some of these classes remain algebraically independent. Here we use the summand $D(n)$ that splits off $MTO(n)$ to show that there are "linear" relations corresponding to elements of $H^*(M(n))$, and that in the case of dimension 2, these relations together with the ones derived from the action of top Steerond squares are the only relations. More precisely, we will show:

Theorem 2.6. i). In $H^*(\Omega_0^\infty MTO(n))$, we have relations $(\Omega^\infty p_n)^*(\sigma^{\infty*}(x)) = 0$ for $x \in H^*(M(n)) \subset H^*(BO(n))$.

ii). For $n = 2$, the only relations among $\mu_{i,j}$'s are the relations above, and $\mu_{2i,2j} = \mu_{i,j}^2$.

iii). Again for $n = 2$, the subalgebra of universally defined characteristic classes in $H^*(\Omega_0^\infty MTO(2))$ is the polynomial algebra generated by $\nu_{i,j}$'s with ij odd, where $\nu_{i,j}$ is defined in [4].

We will give more precise description of the inclusion $H^*(M(n)) \subset H^*(BO(n))$ later.

3 Maps from $MTO(n)$ to $\Sigma^{-n} D(n)$

In this section we use results from [5] to construct maps from $\Sigma^n MTO(n)$'s to $D(n)$'s that form a map of filtered spectra. First of all, we recall results we will need.

3.1 Exact sequences of spectra and the Whitehead conjecture

We start with a definition.

- Definition 3.1** ([5]). *i). A fibration sequence of spectra $F \rightarrow X \xrightarrow{f} Y$ is called exact if there exists a map $g : \Omega^\infty Y \rightarrow \Omega^\infty X$ such that $\Omega^\infty f \circ g \simeq id$.*
- ii). A sequence of spectra $\cdots \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow E_{-1}$ is called exact if for all n , $E_n \rightarrow X_n \rightarrow E_{n-1}$ is exact, where E_n is inductively defined as the fiber of the map $X_n \rightarrow E_{n-1}$.*

The category of spectra being a triangulated category instead of an abelian category, we have some complication here. The notion of exactness with three terms is more or less a counterpart of a split short exactness in abelian categories. The use of this seemingly too strong condition is motivated by the following fact. By definition, an exact sequence of spectra yields an exact sequence of abelian groups upon applying $[Y, -]$ for a suspension spectrum Y , or a spectrum that is a summand of a suspension spectrum. Thus one can regard suspension spectra as free objects, summands of suspension spectra as projective objects, and carry out homological algebra in the category of spectra.

Now, one of the main results of [5], reads as follows.

Theorem 3.2 ([5, Theorem 1.1]). *Let d_k be defined by the composition*

$$d_k : M(k+1) \hookrightarrow BT(k+1)_+ = B(T(1) \times T(k))_+ \xrightarrow{tr} BT(k)_+ \rightarrow M(k)$$

where tr is the Becker-Gottlieb transfer, and the first and last map are obtained via the splitting $M(k) \simeq e'_k BT(k)_+$ with the conjugate Steinberg idempotent e'_k . Then the sequence

$$\cdots \xrightarrow{d_{k+1}} M(k+1) \xrightarrow{d_k} M(k) \rightarrow \cdots \rightarrow M(1) \xrightarrow{d_0} M(0) \xrightarrow{\epsilon} H\mathbb{Z}/2 \quad (7)$$

is exact.

We note that in [5], the sequence above is shown to be equivalent to another sequence consisting of $M(k+1)$'s, whose exactness is known as the mod 2 Whitehead conjecture (Corollary 1.2 *loc.cit.*).

3.2 Complexes of spectra

We still need some more definitions.

- Definition 3.3.** *i). By a chain complex of spectra (E_n, d_n) we understand a sequence of spectra E_n with maps $d_{n-1} : E_n \rightarrow E_{n-1}$ so that the composition $E_{n+1} \rightarrow E_n \rightarrow E_{n-1}$ is null for all n . By a map f of chain complexes of spectra $(E_n, d_E) \rightarrow (F_n, d_F)$ we mean a collection of maps $f_n : E_n \rightarrow F_n$ such that $f_n d_E = d_F f_{n+1}$. Two complexes are said to be isomorphic if there are maps f from E to F such that all f_n 's are homotopy equivalences.*
- ii). Let F_*X be a filtered spectrum. Define its associated graded complex $Gr_\bullet(F_*X)$ by $Gr_0(F_*X) = F_0X$, $Gr_i(F_*X) = \Sigma^{-i} cofib(F_{i-1}X \rightarrow F_iX)$, with obvious maps $Gr_i F_*X \rightarrow Gr_{i-1} F_*X$.*

Remark 3.4. *It follows from an easy diagram chasing that two filtered spectra are equivalent if and only if their associated graded complexes are isomorphic. Thus Gr provides an embedding of the category of filtered spectra into that of complexes of spectra.*

Example 3.5. *i). Let $F_n X = D(n)$. Then the associated graded complex $Gr_\bullet(F_* X)$ is*

$$\cdots \rightarrow M(n+1) \xrightarrow{\delta_n} M(n) \rightarrow \cdots \rightarrow M(0)$$

considered in [5, Corollary 1.2]. This complex, with augmentation to $H\mathbb{Z}/2$ added, was shown to be equivalent with the complex 7 in [5, section 6].

ii). Let $F_n Y = \Sigma^n MTO(n)$. Then the associated graded complex $Gr_\bullet(F_ X)$ is given by $(BO(n)_+, tr)$ where tr is the Becker-Gottlieb transfer associated to the inclusion $O(n-1) \subset O(n)$, as the Becker-Gottlieb transfer $BO(n) \rightarrow BO(n-1)$ factors as $BO(n)_+ \rightarrow MTO(n-1) \rightarrow BO(n-1)_+$. We can also see that $(BO(n)_+, tr)$ is a complex directly by noting that as $O(n) \subset O(n) \times O(2) \subset O(n+2)$, thus $S^1 \cong \{1\} \times SO(2) \subset O(n) \times O(2)$ normalizes $O(n) \subset O(n+2)$, so the transfer associated to $O(n) \subset O(n+2)$ is trivial. Moreover, the complex of free spectra $(BO(n)_+, tr)$ is augmented over $H\mathbb{Z}/2$ since the composition $BO(1)_+ \rightarrow BO(0)_+ \rightarrow H\mathbb{Z}/2$ is trivial. This is just another way of saying that the transfer in $\mathbb{Z}/2$ -cohomology $H^*(BO(0)_+; \mathbb{Z}/2) \rightarrow H^*(BO(1)_+; \mathbb{Z}/2)$ is trivial.*

Now we are ready to do homological algebra. We recall the following.

Proposition 3.6. *Let (P_\bullet, d_\bullet) be a cochain complex of projective R -modules with an augmentation $P_0 \rightarrow A$, and (A_\bullet, d_\bullet) be a resolution of A . Then we get a cochain map from (P_\bullet, d_\bullet) to (A_\bullet, d_\bullet) .*

This is an easy exercise using the definition of projectives and exactness, and proof is omitted. We now translate this to our setting.

Proposition 3.7. *Let (P_\bullet, d_\bullet) be a chain complex of projective spectra with an augmentation $P_0 \rightarrow A$, and (A_\bullet, d_\bullet) be a resolution of A . Then we get a map of chain complexes from (P_\bullet, d_\bullet) to (A_\bullet, d_\bullet) .*

Proof. First, note that Proposition 3.6 is proved using the fact that if P is projective and $C_0 \rightarrow C_1 \rightarrow C_2$ is a short exact sequence, then $Hom(P, C_0) \rightarrow Hom(P, C_1) \rightarrow Hom(P, C_2)$ is short exact. Thus it suffices to show that if P is a suspension spectrum (or a summand of a suspension spectrum, but this doesn't affect anything), and $C_0 \rightarrow C_1 \rightarrow C_2$ is a short exact sequence of spectra, then $[P, C_0] \rightarrow [P, C_1] \rightarrow [P, C_2]$ is a short exact sequence of abelian groups. Write $P = \Sigma^\infty X$. Then we have $[P, C_i] \cong [X, \Omega^\infty C_i]$. By definition of the short exactness, the map $\Omega^\infty C_1 \rightarrow \Omega^\infty C_2$ (as well as loop on this) admits a section, thus $[P, C_1] \cong [X, \Omega^\infty C_1] \rightarrow [X, \Omega^\infty C_2] \cong [P, C_2]$ is (split-)epi. Since $C_0 \rightarrow C_1 \rightarrow C_2$ is a cofibration of spectra, it follows that $[P, C_0] \rightarrow [P, C_1] \rightarrow [P, C_2]$ is a short exact. \square

3.3 Construction of the maps

By the examples of previous section, and Proposition 3.7, we have a maps of chain complexes of spectra $(BO(n)_+, tr) \rightarrow (M(n), d_n)$ That is, we have proven the existence of maps f_n such that the following square commutes

$$\begin{array}{ccc} BO(n)_+ & \xrightarrow{tr} & BO(n-1)_+ \\ \downarrow f_n & & \downarrow f_{n-1} \\ M(n) & \xrightarrow{d_{n-1}} & M(n-1) \end{array}$$

Remark 3.8. The spectrum $M(0)$ is just S^0 , and $M(1) = BO(1)_+$. The maps $BO(0)_+ \rightarrow H\mathbb{Z}/2$ and $\epsilon : M(0) \rightarrow H\mathbb{Z}/2$ coincide with the unit of $H\mathbb{Z}/2$, and the maps f_1 and f_0 can be taken to be the identity.

Now we are ready to prove the following.

Theorem 3.9. Fix maps $f_n : (BO(n)_+, tr) \rightarrow (M(n), d_n)$. Then there exists a map $\alpha_n : MTO(n) \rightarrow \Sigma^{-n}D(n)$ which makes the following diagram commutative for each n

$$\begin{array}{ccc} MTO(n) & \longrightarrow & BO(n)_+ \\ \alpha_n \downarrow & & \downarrow f_n \\ \Sigma^{-n}D(n) & \longrightarrow & M(n). \end{array}$$

Proof. We proceed by induction on n . The case $n = 0$ is trivial. Suppose that we have constructed such α_{n-1} . Consider the following diagram.

$$\begin{array}{ccc} BO(n)_+ & \longrightarrow & MTO(n-1) \\ f_n \downarrow & & \downarrow \alpha_{n-1} \\ M(n) & \longrightarrow & \Sigma^{1-n}D(n-1) \end{array}$$

If we can show that this diagram commutes, then we can define the map α_n using the cofibrations (1) and (5), which will conclude the proof. Note that the two horizontal maps induces trivial maps in cohomology, which implies that the two compositions from the top left corner to bottom right corner factor through E_{n-1} . Here E_n is the spectrum defined in [5], in other words, E_n is the fiber of the map $\Sigma^n D(n) \rightarrow \Sigma^n H\mathbb{Z}/2$. Thus we need to show that the two elements in $[BO(n)_+, E_{n-1}]$ agree. However, by [5, Theorem 1], $[BO(n)_+, E_{n-1}]$ injects to $[BO(n)_+, M(n-1)]$. Thus it suffices to show that the two maps agree after composition with the map $E_{n-1} \rightarrow \Sigma^{1-n}D(n-1) \rightarrow M(n-1)$. Now, consider the following diagram.

$$\begin{array}{ccccc} BO(n)_+ & \longrightarrow & MTO(n-1) & \longrightarrow & BO(n-1)_+ \\ f_n \downarrow & & \downarrow \alpha_{n-1} & & \downarrow f_{n-1} \\ M(n) & \longrightarrow & \Sigma^{1-n}D(n-1) & \longrightarrow & M(n-1) \end{array}$$

The right square is commutative by inductive hypothesis. But we chose our maps f_n so that the big square commutes. This finishes the proof. \square

Note that by construction, the maps α_n form a map of filtered spectra.

Remark 3.10. A reader familiar with [5] must have noticed that our arguments are slightly upside-down. If one follows the proofs in [5], we get α_{n-1} before f_n . As main interests of the authors of *loc.cit.* were not on $D(n)$'s, some statements that could have been proved in *loc.cit* and that we could have quoted are not there. We chose to quote the statements that could be easily found, instead of details of proofs.

4 The splitting

In this section, we construct a map of filtered spectra from $D(n)$'s to $\Sigma^n MTO(n)$'s and conclude the proof of Theorem 2.2. The main ingredient here is the description of $D(n)$ as a summand of Thom spectrum, which is implicit in [13]

4.1 $D(n)$ as a summand of Thom spectrum

Consider the reduced regular representation ρ_n of $T(n) = O(1)^{\times n}$ and let $M(n)_k$ be the stable summand of the Thom spectrum $BT(n)^{k\rho_n}$ corresponding to the Steinberg idempotent e_n . In particular, $M(n)_0 = M(n)$ defined in previous section. According to Takayasu [13] there is a cofibration of spectra

$$\Sigma^k M(n-1)_{2k+1} \longrightarrow M(n)_k \longrightarrow M(n)_{k+1} \quad (8)$$

where the mapping $M(n)_k \xrightarrow{j_k} M(n)_{k+1}$ is induced by the bundle map $k\rho_n \rightarrow (k+1)\rho_n$. We are interested in the case $k = -1$ where Takayasu's cofibration looks like

$$\Sigma^{-1} M(n-1)_{-1} \xrightarrow{i'_{-1}} M(n)_{-1} \xrightarrow{j_{-1}} M(n)_0. \quad (9)$$

Furthermore, the map i'_{-1} satisfies the property $i'_{-1} \circ j_{-2} = i_{-1}$, where $j_{-2} : \Sigma^{-1} M(n-1)_{-2} \rightarrow M(n)_{-1}$ is induced by the inclusion $T(n-1) \subset T(n)$ [13, Theorem A]. We record the following observation which is implicit in [13]

Lemma 4.1. *There is a homotopy equivalence $M(n)_{-1} \rightarrow \Sigma^{-n} D(n)$, i.e. $\Sigma^{-n} D(n)$ is a stable summand of $BT(n)^{-\rho_n}$ corresponding to the Steinberg idempotent.*

Proof. By [13, Proposition 4.1.6], we have $H^*(M(n)_{-1}) \cong H^*(D(n))$. By Corollary 4.2.3 *loc.cit.*, j_{-1}^* is monomorphism, so by the long exact sequence for the cofibration (9) i'_{-1}^* is epimorphism, and the filtered spectrum

$$M(0)_{-1} \rightarrow \cdots \rightarrow \Sigma^{n-1} M(n-1)_{-1} \rightarrow \Sigma^n M(n)_{-1} \rightarrow \cdots$$

realizes the length filtration of the Steenrod algebra. Thus by [3, Corollary 1.4.1] is equivalent to

$$D(0) \rightarrow \cdots \rightarrow D(n) \rightarrow D(n+1) \rightarrow \cdots$$

□

Remark 4.2. *The above can also be proved by direct cohomology calculation using [13, Proposition 4.1.6] and [9, Theorem 5.8].*

4.2 Maps from $\Sigma^{-n} D(n)$ to $MTO(n)$

Denote by β_n the composition

$$\Sigma^{-n} D(n) \longrightarrow BT(n)^{-\rho_n} \longrightarrow BT(n)^{(-\gamma_1)^{\times n}} \longrightarrow BO(n)^{-\gamma_n} \cong MTO(n)$$

where the map at the middle is induced by the embedding $\gamma_1^{\times n} \subset \rho_n$ (or the embedding of virtual vector bundles $-\rho_n \subset (-\gamma_1)^{\times n}$), and the last map is induced by the inclusion $T(n) \rightarrow O(n)$.

Unlike the maps α_n that are constructed as maps between cofibers, the compatibility between different β_n 's are not immediate from the definition. Thus our first task is to show that they form a map of filtered spectra.

Lemma 4.3. *We have following commutative diagram. Thus the maps β_n 's form a map of filtered spectra.*

$$\begin{array}{ccccc}
D(n-1) & \longrightarrow & D(n) & \longrightarrow & \Sigma^n M(n) \\
\beta_{n-1} \downarrow & & \beta_n \downarrow & & \downarrow \\
\Sigma^{n-1} MTO(n-1) & \longrightarrow & \Sigma^n MTO(n) & \longrightarrow & \Sigma^n BO(n)
\end{array}$$

Proof. First consider the square on the left. Since the target of the two compositions is (-1) -connected, and the fiber of the map $j_{-2} : \Sigma^{n-1} M(n-1)_{-2} \rightarrow \Sigma^{n-1} M(n-1)_{-1} = D(n-1)$ has no cell in positive dimension, we see that it suffices to show that they become homotopic after the composition with j_{-2} . Now, consider the diagram

$$\begin{array}{ccccccc}
D(n-1) & \longrightarrow & \Sigma^{n-1} BT(n-1)^{-\rho_{n-1}} & \longrightarrow & \Sigma^{n-1} BT(n-1)^{-\gamma_1^{n-1}} & \longrightarrow & \Sigma^{n-1} BO(n-1)^{-\gamma_{n-1}} \\
\uparrow j_{-2} & & \uparrow & & \parallel & & \parallel \\
\Sigma^{n-1} M(n-1)_{-2} & \longrightarrow & \Sigma^{n-1} BT(n-1)^{-2\rho_{n-1}} & \longrightarrow & \Sigma^{n-1} BT(n-1)^{-\gamma_1^{n-1}} & \longrightarrow & \Sigma^{n-1} BO(n-1)^{-\gamma_{n-1}} \\
\downarrow i_{-1} & & \downarrow & & \downarrow & & \downarrow \\
D(n) & \longrightarrow & \Sigma^n BT(n)^{-\rho_n} & \longrightarrow & \Sigma^n BT(n)^{-\gamma_1^n} & \longrightarrow & \Sigma^n BO(n)^{-\gamma_n}
\end{array}$$

Everything except on the left commutes by naturality of Thom spectra construction. The top left square commutes because the map on the right is $GL_{n-1}(\mathbb{Z}/2)$ -equivariant. The bottom left square commutes because of the equality $e_{n-1}e_n = e_n$ ([5, Corollary 2.6 (2)], this also follows easily from [9, Proposition 2.5]) and the fact that the map on the right is $GL_{n-1}(\mathbb{Z}/2)$ -equivariant. By [13, Theorem A], the map $i'_{-1} : D(n-1) \rightarrow D(n)$ satisfies the property $i'_{-1} \circ j_{-2} = i_{-1}$, so the conclusion follows. A similar but easier argument applies to the map $D(n) \rightarrow \Sigma^n M(n)$. \square

4.3 Proof of the splitting

So far, we have defined the maps α_n and β_n so that the following diagram commutes.

$$\begin{array}{ccccccc}
S^0 = D(0) & \longrightarrow & D(1) & \longrightarrow & \cdots & \longrightarrow & D(n) & \longrightarrow & \cdots \\
\downarrow \beta_0 & & \downarrow \beta_1 & & & & \downarrow \beta_n & & \\
S^0 = MTO(0) & \longrightarrow & \Sigma MTO(1) & \longrightarrow & \cdots & \longrightarrow & \Sigma^n MTO(n) & \longrightarrow & \cdots \\
\downarrow \alpha_0 & & \downarrow \alpha_1 & & & & \downarrow \alpha_n & & \\
S^0 = D(0) & \longrightarrow & D(1) & \longrightarrow & \cdots & \longrightarrow & D(n) & \longrightarrow & \cdots
\end{array}$$

Now, consider $H^*(\alpha_n \circ \beta_n)$. As $H^*(D(n))$ is monogenic over the Steenrod algebra [9, Proposition 4.3], this map is determined by its image on the bottom class. However, the bottom

cell of $D(n)$ is just the image of $D(0)$, so the bottom class is detected by the pull-back to $H^0(D(0))$. The commutativity of the above diagram then implies that $H^0(\alpha_n \circ \beta_n)$ restricted to $H^0(D(0))$ is identity. Thus $H^*(\alpha_n \circ \beta_n)$ is identity, which concludes the proof of Theorem 2.2

4.4 Further refinements

We have shown in [4] that $BSO(2n+1)_+$ splits off $MTO(2n)$. One may ask how this splitting interacts with the splitting of current paper. We show that they are complementary.

Corollary 4.4. $\Sigma^{-2n}D(2n) \vee BSO(2n+1)_+$ splits off $MTO(2n)$. When $n = 1$, we have homotopy equivalence $MTO(2) \cong \Sigma^{-2}D(2) \vee BSO(3)_+$.

Proof. Denote by f_{2n} the inclusion $O(2n) \subset SO(2n+1)$ given by $A \mapsto \det(A)(A \oplus 1)$, and by $Tr_{f_{2n}}$ the associated Miller-Mann-Mann transfer $BSO(2n+1)_+ \rightarrow MTO(2n)$. Consider the composition

$$(\alpha_{2n} \vee Bf_{2n} \circ p_{2n})^* \circ (\beta_{2n} \vee Tr_{f_{2n}})^* : H^*(BSO(2n+1)) \oplus H^*(\Sigma^{-2n}D(2n)) \rightarrow H^*(BSO(2n+1)) \oplus H^*(\Sigma^{-2n}D(2n)).$$

The components $H^*(BSO(2n+1)) \rightarrow H^*(BSO(2n+1))$ and $H^*(\Sigma^{-2n}D(2n)) \rightarrow H^*(\Sigma^{-2n}D(2n))$ are automorphisms. Consider now the component $H^*(\Sigma^{-2n}D(2n)) \rightarrow H^*(BSO(2n+1))$. This is trivial since the source is generated over the Steenrod algebra by a negative-degree elements, and the target is concentrated in non-negative degrees. Thus the map $(\alpha_{2n} \vee Bf_{2n} \circ p_{2n})^* \circ (\beta_{2n} \vee Tr_{f_{2n}})^*$ is an automorphism. This proves the splitting for general n . When $n = 1$, it suffices to compare the cohomology of both sides, or alternatively, by comparing the fibrations $MTO(2) \rightarrow BO(2)_+ \rightarrow MTO(1)$ and $\Sigma^{-2}D(2) \rightarrow M(2) \rightarrow D(1)$, noting that $BO(2)_+ \cong M(2) \vee BSO(3)_+$ (c.f. [10, Theorem C]), we see that $(\alpha_{2n} \vee Bf_{2n} \circ p_{2n})^*$ induces mod 2 homology equivalence. Since everything in sight is of finite type, this implies that we have a 2-local homotopy equivalence. \square

5 Homology of the associated infinite loop spaces

In this section, we discuss the consequences of our splitting theorem to the homology of associated infinite loop spaces.

5.1 Exact sequences

We start with generalities on summands of suspension spectra.

Lemma 5.1. Let M be a spectrum such that M splits off $\Sigma^\infty X$ where X is a space. Denote B a basis of $\tilde{H}_*(M)$. Then there are elements $s_x \in H_*(\Omega^\infty M)$ such that

$$H_*(\Omega^\infty M) = \mathbb{Z}/2[Q^I(s_x); \text{excess}(I) > |x|, I \text{ allowable}], \sigma_*^\infty(s_x) = x,$$

where Q^I denotes the Dyer-Lashof operation ([6]). Suppose further that the splitting of M is obtained by an idempotent of the form $f = \sum_i f_i$, where f_i 's are self-maps of the space X . Then we can choose s_x in such a way that in $H_*(QX)$ we have $s_x = x$ modulo decomposables in $H_*(QX)$, where we identify $H_*(X)$ with its image in $H_*(QX)$ via the canonical map $X \rightarrow QX$, and $H_*(\Omega^\infty X)$ with its image in $H_*(QX)$ via the splitting map.

Proof. The first statement is straightforward, so its proof is omitted. The second statement follows as we have

$$H_*(\Omega^\infty(f)) = H_*(\Omega^\infty(\Sigma_i f_i)) \equiv \Sigma_i H_*(\Omega^\infty(f_i))$$

modulo decomposables. \square

Thus $H_*(\Omega^\infty M(n))$'s are polynomial Hopf algebra. As a matter of fact they are bipolynomial, but we will not need this. A nice feature of such Hopf algebras is that a short exact sequence involving only these Hopf algebras always splits as algebras, thus inducing a short exact sequence of indecomposables. Besides, the map showing up in such a short exact sequence can be effectively studied by studying the induced map in the indecomposables, with little loss of information. All these considerations lead to the following refinement of Proposition 2.4

Proposition 5.2. *The following sequence of Hopf algebra is exact. It gives rise to an exact sequence of graded vector spaces after taking the module of indecomposables.*

$$\cdots \rightarrow H_*(\Omega^\infty M(n)) \rightarrow H_*(\Omega^\infty M(n-1)) \rightarrow \cdots H_*(\Omega^\infty M(2)) \rightarrow H_*(\Omega_0^\infty B\mathbb{Z}/2_+) \rightarrow H_*(Q_0 S^0) \rightarrow \mathbb{Z}/2$$

Furthermore the image of $H_*(\Omega^\infty M(n)) \rightarrow H_*(\Omega^\infty M(n-1))$ is isomorphic to $H_*(\Omega_0^\infty D(n-1))$.

Proof. Denote by $D'(n)$ the fiber of the map $\Sigma^{-n}D(n) \rightarrow \Sigma^{-n}H\mathbb{Z}/2$ corresponding to the bottom class. Then by theorem 3.2, we see that the fibration $D'(n) \rightarrow M(n) \rightarrow D'(n-1)$ is exact. Note that $\Omega^\infty D'(n) \cong \Omega^\infty \Sigma^{-n}D(n)$ for $n \geq 1$, and $\Omega^\infty D'(0) \cong Q_0 S^0$. By definition a three-term exact sequence of spectra leads to a short exact sequence of Hopf algebras by applying $H_*(\Omega^\infty(-); \mathbb{Z}/2)$. Splicing together the short exact sequence thus obtained, we get an exact sequence as in the statement of Proposition, except the last entries which are

$$\cdots H_*(\Omega^\infty B\mathbb{Z}/2_+) \rightarrow H_*(Q_0 S^0) \rightarrow \mathbb{Z}/2[\mathbb{Z}/2] \rightarrow \mathbb{Z}/2.$$

Noting that we have $H_*(\Omega^\infty X_+) \cong H_*(\Omega_0^\infty X_+) \otimes \mathbb{Z}/2[\mathbb{Z}]$ for connected X , in particular $X = B\mathbb{Z}/2$ and $X = pt$, and that $\mathbb{Z}/2[\mathbb{Z}] \rightarrow \mathbb{Z}/2[\mathbb{Z}] \rightarrow \mathbb{Z}/2[\mathbb{Z}/2]$ is exact, we see that the sequence in the Proposition is exact. As everything insight is polynomial, they remain exact after passing to indecomposables. \square

Remark 5.3. *The $Gl_n(\mathbb{Z}/2)$ action on $BT(n)_+$ extends that on $BT(n)$. Thus it is easy to see from the definition of e'_n ([9]) that we have $e'_n BT(n) = e'_n BT(n)_+$ for $n > 1$. Thus $\Omega_0^\infty M(n) = \Omega^\infty M(n)$ for $n > 1$, and the above exact sequence can be expressed entirely in terms of $\Omega_0^\infty M(n)$'s.*

An immediate consequence is

Corollary 5.4. *$H^*(\Omega_0^\infty MTO(2))$ is a polynomial algebra.*

Proof. By Corollary 4.4 we have $\Omega_0^\infty MTO(2) \cong Q_0 BSO(3)_+ \times \Omega^\infty D'(2)$, noting that $\pi_0(D'(2)) = 0$ since it is a direct factor of $\pi_0(M(2))$. The short exact sequence above implies that $H^*(\Omega^\infty D'(2))$ injects to $H^*(\Omega^\infty M(3))$. Since $M(3)$ is a stable summand of $BO(3)$, we see that $H^*(\Omega^\infty D'(2))$ injects to $H^*(Q_0 BO(3))$ which is polynomial ([16, Theorem 3.11]). Since $H^*(\Omega^\infty D'(2))$ is a connected Hopf algebra, by the structure theorem of Hopf algebras over $\mathbb{Z}/2$ ([1, Theorem 6.1] or [7, Theorem 7.11]), this implies that $H^*(\Omega^\infty D'(2))$ itself is a polynomial algebra. Now the Corollary follows as the other factor $H^*(Q_0 BO(3))$ is polynomial again by [16, Theorem 3.11]. \square

5.2 Relations among μ -classes

As an application of the above, we now prove Theorem 2.6.

Definition 5.5. Define the weight on elements of $H_*(QX)$ by $w(Q^I x) = 2^{l(I)}$ for $x \in \text{Im}(H_*(X) \rightarrow H_*(QX))$, and extend it by $w(xy) = w(x) + w(y)$. Denote by $W_i(H_*(QX))$ the set of weight i elements. The module of indecomposables $QH_*(QX)$ inherits the weights from $H_*(QX)$, and we have $QH_*(QX) \cong \oplus_i W_{2^i} QH_*(QX)$, one can define similarly the weights on elements of $QH_*(\Omega^\infty M(n))$ via the inclusion $QH_*(\Omega^\infty M(n)) \subset QH_*(QX)$. By the second statement of Lemma 5.1, $QH_*(\Omega^\infty M(n))$ also admits a direct sum decomposition $QH_*(\Omega^\infty M(n)) \cong \oplus_i W_{2^i} QH_*(\Omega^\infty M(n))$. In each case, we say that the elements of $W_i(-)$ are homogeneous of weight i .

With this definition, we can state:

Lemma 5.6. The map $QH_*(\Omega^\infty d_k) : QH_*(\Omega^\infty M(k+1)) \rightarrow QH_*(\Omega^\infty M(k))$ sends homogeneous elements to homogeneous elements, and multiplies the weight by 2.

Proof. This is immediate from [5, Proposition 3.6]. \square

Define a decreasing filtration F_i on $QH_*(\Omega^\infty M(n))$ by $F_i QH_*(\Omega^\infty M(n)) = \oplus_{j \geq i} W_j(QH_*(\Omega^\infty M(n)))$. Then it satisfies the following conditions.

- i). $F_1(H_*(\Omega^\infty M(n))) = H_*(\Omega^\infty M(n))$
- ii). if $x \in F_i$ then $Q^s(x) \in F_{2^i}$

Thus by naturality of the Dyer-Lashof operations, we see that any map of infinite loop spaces between $\Omega^\infty M(n)$'s induce filtration preserving map in homology, even though most of the time they don't send homogeneous elements to homogeneous elements. We also note that the homology suspension σ_*^∞ maps isomorphically F_1/F_2 to $H_*(M(n))$. We will show the following.

Lemma 5.7. The map $H_*(\Omega^\infty d_{n-1})$ induces an injection

$$H_*(M(n)) \cong F_1/F_2(QH_*(\Omega^\infty M(n))) \rightarrow F_2/F_4(QH_*(\Omega^\infty M(n-1))).$$

Proof. By Lemma 5.6 for $k = n-1$ we see that $QH_*(\Omega^\infty d_{n-1})$ induces a map from F_1/F_2 to F_2/F_4 . By applying Lemma 5.6 to the case $k = n$, we see that the image of $QH_*(\Omega^\infty d_n)$ is included in F_2 . Alternatively, one can see this by noting that the map $H_*(d_n) = 0$. Thus by the exactness of Proposition 2.4, $F_1/F_2(QH_*(\Omega^\infty M(n)))$ injects to $F_2(QH_*(\Omega^\infty M(n-1)))$. But as we have $F_1/F_2 \cong W_1$ and by Lemma 5.6 W_1 maps to W_2 , we get the desired result. \square

Now we are ready to prove Theorem 2.6. The inclusion $H^*(M(n)) \subset H^*(BO(n))$ is given by $H^*(f_n)$, and this is determined uniquely by its compatibility with $H^*(\alpha_n)$, which in turn is determined uniquely by the fact that $H^{-n}(MTO(n))$ contains only one non-trivial element, and the fact that $H^*(D(n))$ is generated by the bottom class as a module over the Steenrod algebra. The cofibration sequence (2) imply that such a class vanishes if its preimage in $H^*(QBO(n))$ belongs to the image of $H^*(\Omega_0^\infty MTO(n-1))$. Now, Theorem 3.9 implies that we have a commutative diagram

$$\begin{array}{ccccc} BO(n) & \xrightarrow{f_n} & M(n) & & \\ \downarrow & & \downarrow & \searrow d_n & \\ MTO(n-1) & \longrightarrow & \Sigma^{1-n} D(n-1) & \longrightarrow & M(n-1) \end{array}$$

Thus we get

$$\begin{array}{ccccc}
H_*(Q_0BO(n)_+) & \longrightarrow & H_*(\Omega^\infty M(n)) & & \\
\downarrow & & \downarrow & \searrow & \\
H_*(\Omega_0^\infty MTO(n-1)) & \longrightarrow & H_*(\Omega^\infty(\Sigma^{1-n}D(n-1))) & \longrightarrow & H_*(\Omega^\infty M(n-1))
\end{array}$$

Dualizing Lemma above, we see that the space of functionals on $QH_*(\Omega^\infty M(n-1))$ vanishing on F_4 surjects to the space of functionals on $QH_*(\Omega^\infty M(n))$ vanishing on F_2 , which is precisely the image of $\sigma^{\infty*}$. Thus by the commutativity of the diagram above, we see that the image of the composition $PH^*(M(n)) \xrightarrow{\sigma^{\infty*}} PH^*(\Omega^\infty M(n)) \rightarrow PH^*(Q_0BO(n))$ is contained in the image of $PH^*(\Omega_0^\infty MTO(n-1))$. This concludes the proof of i). The other statements follow from [4, Theorem 1.8] and Corollary 4.4.

Remark 5.8. *The formula in [11, Corollary 4.4'] involves a map that sends homogeneous elements of weight 1 to a sum of homogeneous elements of weight 2 and 4. The proof of Proposition 4.5' loc. cit. shows that the terms of weight 4 can be ignored. In the above argument, we show that actually one can use another map which are homogeneous.*

To conclude, we give some explicit examples of those relations. First of all, we have [?, Corollary 3.11]

Proposition 5.9. *The image of $H^*(M(n))$ in $H^*(BO(n))$ is the a free-module over $H^*(BT(n))^{Gl_n(\mathbb{Z}/2)}$ generated by a basis of $A(n-2)Sq^{2^{n-1}, \dots, 2, 1}(x_1^{-1} \dots x_n^{-1})$ where $A(k)$ is the subalgebra of the Steenrod algebra generated by $Sq^1, Sq^2, \dots, Sq^{2^k}$. Here we identify $H^*(BO(n))$ with its image in*

$$H^*(BT(n)) \subset H^*(BT(n))^{-\gamma_n} \cong (x_1 \dots x_n)^{-1} H^*(BT(n))$$

via Bi^* where $i : T(n) \subset O(n)$.

When $n = 2$, $A(0)$ is just the exterior algebra generated by Sq^1 , and $Sq^{2,1}(x_1^{-1}x_2^{-1}) = x_1 + x_2 = \sigma_1$, $Sq^1(\sigma^1) = x_1^2 + x_2^2 = \sigma_1^2$, whereas the Dickson invariants $H^*(BT(n))^{Gl_n(\mathbb{Z}/2)}$ is generated by $w_2 = x_1^2 + x_1x_2 + x_2^2 = \sigma_1^2 + \sigma_2$, $w_3 = x_1x_2(x_1 + x_2) = \sigma_1\sigma_2$, we derive

Corollary 5.10. *The set*

$$\{(\sigma_1^2 + \sigma_2)^i (\sigma_1 \sigma_2)^j \sigma_1^\epsilon; i \geq 0, j \geq 0, \epsilon \in \{1, 2\}\}$$

forms a basis of the image of $H^(M(2))$ in $H^*(BO(2))$.*

Below is a table of these relations in low dimensions.

$$\begin{aligned}
\mu_{1,0} &= 0 \\
\mu_{3,0} + \mu_{1,1} &= 0 \\
\mu_{2,1} &= 0 \\
\mu_{5,0} + \mu_{3,1} + \mu_{1,2} &= 0 \\
\mu_{3,1} &= 0 \\
\mu_{4,1} + \mu_{2,2} &= 0
\end{aligned}$$

Here we have omitted the relations that follow from lower degree relations and the general relation $\mu_{2i,2j} = \mu_{i,j}^2$.

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